TORIC ARC SCHEMES AND QUANTUM COHOMOLOGY OF TORIC VARIETIES.

SERGEY ARKHIPOV AND MIKHAIL KAPRANOV

1. Introduction

This paper is a part of a larger project devoted to the study of Floer cohomology in algebro-geometic context, as a natural cohomology theory defined on a certain class of ind-schemes. Among these ind-schemes are algebro-geometric models of the spaces of free loops.

Let X be a complex projective variety. Heuristically, $HQ(X)$, the quantum cohomology of X, is a version of the Floer cohomology of the universal cover of the free loop space $\mathcal{L}X = \mathrm{Map}(S^1, X)$. However, truly infinite-dimensional approaches to HQ are few. One of the most interesting is the unpublished result of D. Peterson: for a reductive group G the ring $HQ(G/B)$ is identified with the usual topological cohomology *ring* of $\mathcal{F}$, the affine flag variety of G. To be precise, if $A = H_2(G/B)$ is the coweight lattice, then $H^\bullet(\mathcal{F})$ is naturally an algebra over the semigroup ring $\mathbb{C}[A_+]$ of the dominant cone in A and

$$HQ(G/B) = H^\bullet(\mathcal{F}) \otimes_{\mathbb{C}[A_+]} \mathbb{C}[A].$$

The ind-scheme $\mathcal{F}$ has the homotopy type of $\mathcal{L}G$. Peterson's approach was deemed mysterious as the analog of $\mathcal{F}$ for varieties not of the form G/P was not clear.

In the present paper we consider the case when X is a toric Fano variety. It turns out that the concept of arc schemes (see [7] [12]) provides a replacement of the space $\mathcal{F}$ in this situation. The main result of the paper, Theorem 4.7, identifies $HQ(X)$ with an appropriate localization of the topological cohomology of $\Lambda^0 X$, the toric version for the arc scheme. The action of the generators q arises from the multiplication of arcs by the parameter t. For example, when $X = P^{N-1}$, the scheme $\Lambda^0 X$ is the infinite-dimensional schematic projective space

$$P(\mathbb{C}^N[[t]]) = \mathrm{Proj}\ \mathbb{C}[z_{i,n},\ i = 1, ..., N,\ n \geq 0],$$

where the $z_{i,n}$ are the coefficients of N indeterminate formal power series $z_i(t) = \sum_{n=0}^\infty z_{i,n} t^n$. Its topological cohomology ring is $\mathbb{C}[q]$ where q is the hyperplane class. The simultaneous multiplication of the series by t embeds this scheme into itself as a projective subspace of complex codimension N, giving the familiar relation $q = x^N$ in $HQ(P^{N-1})$.

The scheme $\Lambda^0 X$ is a part of an ind-scheme ΛX with an action by $H_2(X, \mathbb{Z})$ which can be considered as an algebro-geometric model of the universal cover of $\mathcal{L}X$. For

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example, for $X = P^{N-1}$

$$\Lambda(P^{N-1}) = P\big(\mathbb{C}^N((t))\big) = \text{“}\varinjlim\text{“}_m \operatorname{Proj} \mathbb{C}[z_{i,n}, i = 1, ..., N,\ n \geq -m].$$

Note that a similar model was constructed by Y. Vlassopoulos [14] (see also [11]) who developed the approach of Givental [9]. The Vlassopoulos model $V(X)$ is a union of finite-dimensional varieties while ΛX is not. For example, for $X = P^{N-1}$ we have

$$V(P^{N-1}) = P(\mathbb{C}^N[z, z^{-1}]) = \text{“}\varinjlim\text{“}_m \text{“}\varinjlim\text{“}_M \operatorname{Proj} \mathbb{C}[z_{i,n}, i = 1, ..., N,\ m \leq n \leq M].$$

Our space ΛX, constructed as it is from infinite-dimensional schemes, is locally an ind-pro-object in the category of finite-dimensional varieties. The duality between the ind- and pro-directions is essential in our approach to Floer cohomology. For example, in the study of the space $V(X)$ in *loc. cit.* the indices m and M are treated differently while from the point of view of $\mathbb{C}[z, z^{-1}]$ they are completely symmetric. In our approach this difference is built in the definition of an ind-scheme.

The idea of modeling the loop spaces by considering formal Laurent series as opposed to Laurent polynomials was advocated for a long time by V. Drinfeld. In fact, our ind-scheme ΛX is an example of a T-smooth ind-scheme in the sense of [8]. We would like to thank H. Iritani for bringing our attention to his work and to the papers of Givental and Vlassopoulos.

2. Quantum cohomology of toric varieties (after Batyrev).

2.1. **Combinatorics of toric varieties.** Let X be a smooth compact toric variety defined over $\mathbb{C}$. From now on we fix the realization of X as a quotient, following M. Audin and D. Cox, see [1], [5]. More precisely, we represent $X = (\mathbb{C}^N - \mathcal{D})/S$ where S is a torus acting diagonally on $\mathbb{C}^N$ (with coordinates $z_1, \ldots, z_N$) and $\mathcal{D} \subset \mathbb{C}^N$ is a closed subscheme called the exceptional locus. It has the following structure (see [2] for more detail). Denote the coordinate hyperplane $\{z_i = 0\} \subset \mathbb{C}^N$ by V_i. For $I \subset \{1, \ldots, N\}$ denote $V_I := \bigcap_{i \in I} V_i$. The set $\mathcal{D}$ has the form

$$\mathcal{D} = \bigcup_{I \in \mathcal{P}} V_I$$

where $\mathcal{P} \subset 2^{\{1,\ldots,N\}}$ is a certain subset whose elements are called primitive collections. It is known that S acts freely on $\mathbb{C}^N - \mathcal{D}$ and X is the categorical quotient of $\mathbb{C}^N - \mathcal{D}$ by S.

We denote by A (resp. by B) the lattice of 1-parametric subgroups in S (resp., the lattice of characters of S). In other words,

$$A = \operatorname{Hom}(\mathbb{C}^*, S),\ B = \operatorname{Hom}(S, \mathbb{C}^*).$$

By construction we have

$$B \xrightarrow{\sim} \operatorname{Pic}(X) = H^2(X, \mathbb{Z}),\ A = B^\vee \xrightarrow{\sim} H_2(X, \mathbb{Z}).$$

Below we denote the image of $V_i - \mathcal{D}$ in X by Z_i. Notice that Z_i is a divisor in X. Consider the class of Z_i in $\operatorname{Pic}(X)$ denoted by $[Z_i]$. In terms of the isomorphism

$\mathrm{Pic}(X) = \mathrm{Hom}(S, \mathbb{C}^*)$ the class $[Z_i]$ corresponds to the character of S as follows:

$$S \xrightarrow{\text{action}} (\mathbb{C}^*)^N \xrightarrow{p_i} \mathbb{C}^*,$$

where p_i denotes the projection to the i-th factor.

2.2. **Combinatorial description for cohomology of X.** The following statement is the combinatorial description of the $\mathbb{C}$-algebra $H^\bullet(X, \mathbb{C})$.

Proposition 2.3. *The $\mathbb{C}$-algebra $H^\bullet(X,\mathbb{C})$ has generators $[Z_1], \dots, [Z_N] \in H^2(X,\mathbb{C})$ and the relations as follows:*

(i) *Linear relations among the $[Z_i]$ in $H^2(X,\mathbb{C})$.*
(ii) *For any $I \in \mathcal{P}$ we have $\prod\limits_{i\in I} [Z_i] = 0$.* □

2.4. **Combinatorial description for quantum cohomology of X (after Batyrev).** Below we present the main result of Batyrev (see [2], Theorem 9.5) as made more precise in [6], Example 11.2.5.2, providing the combinatorial description of the quantum cohomology algebra for a toric Fano variety X.

Namely, as a vector space, the quantum cohomology algebra for X denoted by $HQ(X)$ equals *by definition* $H^\bullet(X,\mathbb{C}) \otimes \mathbb{C}[A]$. Here $\mathbb{C}[A]$ denotes the group algebra of the lattice A formed by Laurent polynomials $\sum\limits_{a\in a} c_a q^a$.

The multiplication on $HQ(X)$ as on a $\mathbb{C}[A]$-algebra is defined by generators and relations as follows.

The diagonal action of S on $\mathbb{C}^N$ defines a homomorphism

$$\beta : A \to \mathbb{Z}^N = \mathrm{Hom}(\mathbb{C}^*, (\mathbb{C}^*)^N).$$

Let $A_+ = \beta^{-1}(\mathbb{Z}^N_+)$.

Proposition 2.5. *Assume that X is a Fano variety. The $\mathbb{C}[A]$-algebra $HQ(X)$ has generators $[\mathcal{Z}_1], \dots, [\mathcal{Z}_N]$ corresponding to the cocycles $[Z_1], \dots, [Z_N] \in H^2(X,\mathbb{C})$ and the relations as follows:*

$(\mathrm{i})_q$ *The generators $[\mathcal{Z}]_i$ satisfy the same linear relations as the elements*

$$[Z_1], \dots, [Z_N] \in H^2(X,\mathbb{C}).$$

$(\mathrm{ii})'_q$ *For every $a \in A_+$ we have*

$$\prod_{i=1}^{N} [\mathcal{Z}_i]^{\beta_i(a)} = q^a. \quad \square$$

Remark: In particular, it follows that the $\mathbb{C}[A]$-algebra defined by the relations $(\mathrm{i})_q$ and $(\mathrm{ii})'_q$ is free as a $\mathbb{C}[A]$-module and has rank equal to dim $H^\bullet(X,\mathbb{C})$.

3. Cohomology of the toric arcs space.

Recall (see [7], [12]) that for every $\mathbb{C}$-scheme Y we have the scheme L^0Y of formal arcs, representing the following functor:

$$\mathrm{Hom}_{\mathsf{Sch}}(U, L^0Y) = \mathrm{Hom}_{\mathsf{LRS}}((U, \mathcal{O}_U[[t]]), (Y, \mathcal{O}_Y))$$

where LRS is the category of locally ringed spaces. For $Y = \mathrm{Spec}(A)$ affine and $U = \mathrm{Spec}(R)$ we have

$$\mathrm{Hom}_{\mathsf{Sch}}(U, L^0Y) = \mathrm{Hom}_{\mathbb{C}\text{-}\mathsf{Alg}}(A, R[[t]]).$$

It follows that $L^0Y = \mathrm{Spec}\ A^{[[t]]}$ where $A^{[[t]]}$ is generated by symbols $a[n], a \in A, n \in \mathbb{Z}_+$ subject to

$$(ab)[n] = \sum_{i+j=n} a[i]b[j],$$

compare with the formulas of Borisov [3].

In particular, for $Y = \mathbb{C}^N$ with coordinates $z_1, \ldots, z_n$ the set $L^0Y(\mathbb{C})$ is identified with $\mathbb{C}^N[[t]]$, the set of N-tuples of formal Taylor series

$$(z_1(t), \ldots, z_N(t)),\ \ z_i(t) = \sum_{n=0}^{\infty} z_{i,n}t^n.$$

We assume the situation and notation of Section 2.

Proposition 3.1. *The torus S acts freely on the scheme $L^0Y - L^0\mathcal{D}$.* □

Definition 3.2. *The Toric arcs scheme Λ^0X is defined to be the categorical quotient $\Lambda^0X := (L^0Y - L^0\mathcal{D})/S$.*

Remarks: (a) The scheme Λ^0X is not quasicompact, i.e., it cannot be covered by finitely many affine open parts. Its construction is completely analogous to that of toric varieties in Section 2, so it can be considered as an "infinite-dimensional toric variety". In fact, it is acted upon by the group scheme

$$(\mathbb{C}^*)^\infty = \mathrm{Spec}\,\mathbb{C}[z_{i,n}^{\pm 1}].$$

The subgroup S embedded diagonally into $(\mathbb{C}^*)^\infty$, acts trivially, so the role of the torus acting on a toric variety, is played in this case by $(\mathbb{C}^*)^\infty/S$.

(b) Let $\mathbb{A}^\infty := \mathrm{Spec}\,\mathbb{C}[x_1, x_2, \ldots]$. Recall [13] that a scheme Y is called essentially smooth if it is covered by open subsets which are isomorphic to $W \times \mathbb{A}^\infty$ where W is a smooth algebraic variety. In the sequel we will call such schemes simply smooth. The scheme Λ^0X is smooth, in fact it is covered by open subsets isomorphic to $\mathbb{A}^\infty$.

Let $p:\ L^0\mathbb{C}^N \longrightarrow \mathbb{C}^N$ be the canonical projection. We define $\mathcal{V}_i = p^{-1}(V_i) \subset L^0\mathbb{C}^N$ and $\mathcal{Z}_i \subset \Lambda^0X$ to be the image of $\mathcal{V}_i - L^0\mathcal{D} \cap \mathcal{V}_i$.

Following [13], for any $\mathbb{C}$-scheme (possibly, of infinite type) Σ we introduce the complex topology on the set $\Sigma(\mathbb{C})$. In particular we can speak about the topological cohomology $H^\bullet(\Sigma(\mathbb{C}), \mathbb{C})$. The latter space will be shortly denoted by $H^\bullet(\Sigma, \mathbb{C})$.

Proposition 3.3. *Odd cohomology spaces of Λ^0X vanish. The graded ring $H^{2\bullet}(\Lambda^0X, \mathbb{C})$ is isomorphic to the symmetric algebra* $\mathrm{Sym}^\bullet(H^2(X, \mathbb{C}))$.

Proof. We begin the proof with the following statement.

Lemma 3.4. *$H^\bullet(L^0\mathbb{C}^N - L^0\mathcal{D}, \mathbb{C})$ vanishes in positive degrees.*

Proof. Since $\mathcal{D} = \bigcup_{I \in \mathcal{P}} V_I$, we have $L^0\mathcal{D} = \bigcup_{I \in \mathcal{P}} L^0V_I$ and each L^0V_I is an affine subspace in $L^0\mathbb{C}^N$ of infinite codimension. The statement of the Lemma follows immediately. □

Now, in view of the previous Lemma, the Proposition follows from the Serre spectral sequence of the fibration $\rho: \ L^0\mathbb{C}^N - L^0\mathcal{D} \xrightarrow{S} \Lambda^0 X$ and the fact that $H^\bullet(S,\mathbb{C})$ is the exterior algebra of the space $B\otimes\mathbb{C} = H^2(X,\mathbb{C})$. □

Corollary 3.5. *The classes* $[\mathcal{Z}_i] \in H^2(\Lambda^0 X,\mathbb{C})$ *generate* $H^\bullet(\Lambda^0 X,\mathbb{C})$ *as a ring.* □

4. Self-embeddings and Gysin maps.

Let $a \in A_+$. Then the 1-parameter subgroup $t^{\beta(a)}$ of S can be considered as a $\mathbb{C}$-point of $L^0(\mathbb{C}^*)^N$. Using the action of $(\mathbb{C}^*)^N$ on $\mathbb{C}^N$, for any $\mathbb{C}$-point $\gamma(t)$ of $L^0\mathbb{C}^N$ we can form a new $\mathbb{C}$-point $t^{\beta(a)}\gamma(t)$. It is clear that the correspondence $\gamma(t)\mapsto t^{\beta(a)}\gamma(t)$ gives rise to a morphism of schemes $\epsilon_a : \ L^0\mathbb{C}^N \longrightarrow L^0\mathbb{C}^N$ and that ϵ_a is a closed embedding. It is also clear (from the fact that β is a homomorphism) that

$$\epsilon_{a+b} = \epsilon_a \circ \epsilon_b, \ \ a, b \in A_+.$$

We define the subscheme $L^a\mathbb{C}^N \subset L^0\mathbb{C}^N$ to be the image of ϵ_a.

Proposition 4.1. $L^a\mathbb{C}^N$ *is equal to the closed subscheme of* $L^0\mathbb{C}^N$ *formed by* N*-tuples of series* $\gamma(t) = (\gamma_1(t),\dots,\gamma_N(t))$ *such that* $\operatorname{ord}_t \gamma_i(t) \geq \beta_i(a)$. □

Introduce a partial order $\leq$ on A_+ by putting $a \leq b$ iff $b - a \in A_+$. It is clear from the above Proposition that $L^b\mathbb{C}^n \subset L^a\mathbb{C}^N$ if and only if $a \leq b$.

Corollary 4.2. *We have* $\operatorname{codim}_{L^a\mathbb{C}^N} L^b\mathbb{C}^N = \sum_{i=1}^N \beta_i(b-a)$. □

It is clear that for $a \in A_+$ we have $\epsilon_a(L^0\mathbb{C}^N - L^0\mathcal{D}) \subset L^0\mathbb{C}^N - L^0\mathcal{D}$ and that ϵ_a commutes with the action of S. Therefore the map descends to a closed embedding

$$E_a : \ \Lambda^0 X \longrightarrow \Lambda^0 X, \ \ E_{a+b} = E_a \circ E_b.$$

The image of E_a will be denoted by $\Lambda^a X$. We also conclude that $\Lambda^b X \subset \Lambda^a X$ if and only if $a \leq b$ and, if so,

$$\operatorname{codim}_{\Lambda^a X} \Lambda^b X = \sum_{i=1}^N \beta_i(b-a).$$

Proposition 4.3. *The inverse image map* $E_a^* : \ H^\bullet(\Lambda^0 X,\mathbb{C}) \to H^\bullet(\Lambda^0 X,\mathbb{C})$ *is the identity for any* $a \in A_+$.

Proof. The maps ϵ_a and E_a define an embedding of the fibration

$$\rho: \ L^0\mathbb{C}^N - L^0\mathcal{D} \xrightarrow{S} \Lambda^0 X$$

into itself. On the cohomology of the total spaces $E_a^* = \mathrm{Id}$. On the cohomology of fibers E_a^* is identity as well. Our statement follows from functoriality of Serre spectral sequences. □

Let $Y \xrightarrow{q} Z$ be a smooth closed embedding of (possibly infinite-dimensional) schemes of finite codimension d. This implies that locally on the complex topology the embedding of $\mathbb{C}$-points is identified with $Y(\mathbb{C}) \to Y(\mathbb{C})\times\mathbb{C}^d$. This gives an identification in the derived category

$$\underline{R\Gamma}_{Y(\mathbb{C})}(\underline{\mathbb{C}}_{Z(\mathbb{C})}) \to \underline{\mathbb{C}}_{Y(\mathbb{C})}[2d]$$

and hence gives the Gysin maps

$$q_! : H^m(Y, \mathbb{C}) \to H^{m+2d}(Z, \mathbb{C}).$$

Let $\mathbb{C}[A_+]$ be the semigroup algebra of A_+. Its elements are represented as (formal) polynomials $\sum_{a \in A_+} c_a q^a$ in a variable $q \in \mathrm{Hom}(A, \mathbb{C}^*)$. We make the space $H^\bullet(\Lambda^0 X, \mathbb{C})$ into a $\mathbb{C}[A_+]$-module by putting for $\alpha \in H^\bullet(\Lambda^0 X, \mathbb{C})$

$$\left(\sum c_a q^a\right) \cdot \alpha := \sum c_a E_{a!}(\alpha). \tag{1}$$

Here $E_{a!}: \ H^m(\Lambda^0 X, \mathbb{C}) \longrightarrow H^{m+2\sum \beta_i(a)}(\Lambda^0 X, \mathbb{C})$ is the Gysin map corresponding to the smooth closed embedding E_a. The fact that the formula above defines a $\mathbb{C}[A_+]$-module structure follows immediately from the composition rule $E_{a+b} = E_a \circ E_b$ and the next Lemma.

Lemma 4.4. *Let $Y_1 \xrightarrow{q} Y_2 \xrightarrow{p} Y_3$ be smooth closed embeddings of schemes of finite codimension. Then we have the equality of Gysin maps*

$$(pq)_! = p_! \circ q_!. \quad \square$$

Recall the key property of Gysin maps (projection formula). It follows at once from the sheaf-theoretic definition of Gysin maps above.

Proposition 4.5. *Let $q: \ Y \longrightarrow Z$ be a closed embedding of smooth $\mathbb{C}$-schemes of finite codimension d. Then the Gysin map*

$$q_!: \ H^\bullet(Y, \mathbb{C}) \longrightarrow H^\bullet(Z, \mathbb{C})$$

is $H^\bullet(Z, \mathbb{C})$-linear, i.e. we have

$$q_!(q^*(\alpha) \cap \beta) = \alpha \cap q_!(\beta), \ \alpha \in H^\bullet(Z, \mathbb{C}), \ \beta \in H^\bullet(Y, \mathbb{C}). \quad \square$$

Applying this to $q = E_a$, $Y = Z = \Lambda^0 X$ and using Proposition 4.3 we get the following

Corollary 4.6. *The formula 1 makes the ring $H^\bullet(\Lambda^0 X, \mathbb{C})$ into a $\mathbb{C}[A_+]$-algebra.* $\square$

Now we come to the main statement of the paper.

Theorem 4.7. *The localized algebra*

$$H^\bullet(\Lambda^0 X, \mathbb{C}) \otimes_{\mathbb{C}[A_+]} \mathbb{C}[A]$$

is isomorphic the the quantum cohomology algebra $HQ(X, \mathbb{C})$ (isomorphism of $\mathbb{C}[A]$-algebras).

5. Proof of Theorem 4.7.

We construct a homomorphism of $\mathbb{C}[A]$-algebras

$$\Phi: \ HQ(X) \longrightarrow H^\bullet(\Lambda^0 X, \mathbb{C}) \otimes_{\mathbb{C}[A_+]} \mathbb{C}[A]$$

by putting $\Phi([Z_i]) = [\mathcal{Z}_i] \otimes 1$. In order to see that this indeed defines a homomorphism, we need to check the relations $(i)_q$ and $(ii)'_q$ from Proposition 2.5. For $(i)_q$ this follows form the identification

$$H^2(X, \mathbb{C}) \tilde{\to} H^2(\Lambda^0 X, \mathbb{C}) \tilde{\to} B \otimes \mathbb{C}$$

which takes $[Z_i]$ to $[\mathcal{Z}_i]$. It remains to check $(ii)'_q$.

As in Section 3, we introduce coordinates $z_{i,n}$, $i = 1, \ldots, N$ and $n \in \mathbb{Z}_+$, in $L^0\mathbb{C}^N$ so that a $\mathbb{C}$-point has the form

$$\gamma(t) = (\gamma_1(t), \ldots, \gamma_N(t)),\ \ \gamma_i(t) = \sum_{n=0}^{\infty} z_{i,n} t^n.$$

Let $\mathcal{V}_{i,n} \subset L^0\mathbb{C}^N$ be the hypersurface $z_{i,n} = 0$ and $\mathcal{Z}_{i,n} \subset \Lambda^0 X$ be the corresponding divisor. In particular we have $\mathcal{Z}_i = \mathcal{Z}_{i,0}$. On the other hand, $\mathcal{Z}_{i,n}$ is linearly equivalent to $\mathcal{Z}_{i,m}$ for any n, m. This is because the function $z_{i,n}/z_{i,m}$ on $L^0\mathbb{C}^N$ is homogeneous of degree 0 with respect to the action of the torus S and it descends a rational function on $\Lambda^0 X$ with divisor $\mathcal{Z}_{i,n} - \mathcal{Z}_{i,m}$. Therefore we can write the self-intersection $[\mathcal{Z}_i]^r$ as the class of the subvariety $\mathcal{Z}_{i,0} \cap \ldots \cap \mathcal{Z}_{i,r-1}$. So the class $\prod [\mathcal{Z}_i]^{\beta_i(a)}$ in $(ii)'_q$ is the class of the subvariety

$$\bigcap_{i=1}^{N} \mathcal{Z}_{i,0} \cap \ldots \cap \mathcal{Z}_{i,\beta_i(a)-1}$$

which coincides with $\Lambda^a X$ in virtue of Proposition 4.1.

Further, $[\Lambda^a X]$ is equal to the result of action of $q^a = E_{a!}$ on the element $1 \in H^\bullet(\Lambda^0 X, \mathbb{C})$. So the homomorphism Φ is well defined. Notice that it is surjective since $H^\bullet(\Lambda^0 X, \mathbb{C})$ is generated by the classes $[\mathcal{Z}_i]$ as a $\mathbb{C}$-algebra. It remains to prove that Φ is injective.

Proposition 5.1. *The cohomology ring $H^\bullet(\Lambda^0 X, \mathbb{C})$ is torsion free as a $\mathbb{C}[A_+]$-module and has generic rank equal to* $\dim H^\bullet(X, \mathbb{C})$.

Proof. We define a locally closed subscheme $L^{=a}\mathbb{C}^N \subset L^0\mathbb{C}^N$ by

$$L^{=a}\mathbb{C}^N := L^a\mathbb{C}^N - \bigcup_{b>a} L^b\mathbb{C}^N.$$

We also define $\Lambda^{=a}X := \Lambda^a X - \bigcup_{b>a} \Lambda^b X$. For $a \in A_+$ let $I^a :\ \Lambda^{=a}X \hookrightarrow \Lambda^0 X$ be the embedding. Recall that p denotes the canonical projection $L^0\mathbb{C}^N \to \mathbb{C}^N$.

Lemma 5.2. *We have the equalities*

$$L^{=0}\mathbb{C}^N = L^0\mathbb{C}^N - p^{-1}(\mathcal{D}) \text{ and } \Lambda^{=0}X = (L^0\mathbb{C}^N - p^{-1}(\mathcal{D}))/S. \quad \square$$

In particular, $p :\ L^0\mathbb{C}^N \to \mathbb{C}^N$ induces a morphism of schemes $P^0 :\ \Lambda^0 X \longrightarrow X$.

Lemma 5.3. *P^0 is a (Zariski) locally trivial fibration with fiber $\mathbb{A}^\infty$. In particular, the map*

$$(P^0)^* :\ H^\bullet(X, \mathbb{C}) \tilde{\longrightarrow} H^\bullet(\Lambda^{=0}X, \mathbb{C})$$

is an isomorphism of $\mathbb{C}$-algebras. $\square$

Corollary 5.4. *The map*

$$(E_a)^* \circ (P^0)^* : \ H^\bullet(X, \mathbb{C}) \widetilde{\longrightarrow} H^\bullet(\Lambda^{=a} X, \mathbb{C})$$

is an isomorphism of $\mathbb{C}$*-algebras.* □

Consider the Cousin spectral sequence (see [10], p. 227) corresponding to the filtration of $\Lambda^0 X$ by the $\Lambda^a X$ and the constant sheaf $\underline{\mathbb{C}}$ on $\Lambda^0 X$. It has

$$E_1 = \bigoplus_{a \in A_+} H^\bullet_{\Lambda^{=a} X}(\Lambda^0 X, \mathbb{C}), \quad E_\infty = H^\bullet(\Lambda^0 X, \mathbb{C}).$$

Now, since $\Lambda^a X \subset \Lambda^0 X$ is a smooth embedding, we have

$$H^p_{\Lambda^{=a} X}(\Lambda^0 X, \mathbb{C}) = H^{p+2\sum \beta(a_i)}(\Lambda^{=a} X, \mathbb{C}) = H^{p+2\sum \beta(a_i)}(X, \mathbb{C}).$$

In particular, the E_1-term is concentrated in even degrees. It follows that the spectral sequence degenerates and provides a certain filtration on $H^\bullet(\Lambda^0 X, \mathbb{C})$ with the key property:

$$\mathrm{gr}(H^\bullet(\Lambda^0 X, \mathbb{C})) = \mathbb{C}[A_+] \otimes H^\bullet(X, \mathbb{C}).$$

as a $\mathbb{C}[A_+]$-module. Proposition 5.1 is proved. □

We conclude that the $\Phi : \ HQ(X) \to H^\bullet(\Lambda^0 X, \mathbb{C}) \otimes_{\mathbb{C}[A_+]} \mathbb{C}[A]$ is a map from a *free* $\mathbb{C}[A]$-module to a module of the same generic rank (equal to $\dim(H^\bullet(X, \mathbb{C}))$). Moreover, we have already proved that Φ is surjective.

It follows that $H^\bullet(\Lambda^0 X, \mathbb{C}) \otimes_{\mathbb{C}[A_+]} \mathbb{C}[A]$ is a free module. Indeed, freeness of a finitely generated module is equivalent to the fact that the rank of its specialization at any point is the same. In any case, the rank of the specialization is greater or equal than the generic rank.

Now, since the specialization functor is right exact and Φ is surjective, we get that the rank of the specialization is less or equal to the generic rank thus implying the freeness. Now, a surjective morphism between free modules of the same rank should be injective and so an isomorphism.

Theorem 4.7 is proved.

6. Concluding remarks

6.1. **Relation to Floer cohomology.** Let us explain the relation of the above construction with our algebro-geometric approach to Floer cohomology. By inverting the self-embeddings $E_a, a \in A_+$, we include the scheme $\Lambda^0 X$ into an ind-scheme

$$\Lambda X = \text{"}\varinjlim_{a \in A}\text{"} \Lambda^a X$$

on which the group $A = H_2(X, \mathbb{Z})$ acts by automorphisms. This is a toric algebro-geometric model of the universal cover of the free loop space $\mathcal{L}X$. The Floer cohomology of ΛX is defined to be

$$HF(\Lambda X) = \varinjlim_{a \in A} H^{\bullet + 2d(a)}(\Lambda^a X, \mathbb{C}), \quad d(a) = \sum \beta(a_i),$$

where the limit is taken with respect to the Gysin maps. This limit is identified with the extension of scalars from $\mathbb{C}[A_+]$ to $\mathbb{C}[A]$ as in Theorem 4.7.

6.2. **More general varieties.** In a forthcoming paper we plan to generalize the above approach to a large class of smooth projective varieties. Let X be a smooth projective variety over $\mathbb{C}$ together with a choice of a projective embedding, i.e., represented as $X = \mathrm{Proj}(A)$. We define $\Lambda^0 X = \mathrm{Proj}(A^{[[t]]}$ where the algebra $A^{[[t]]}$ is defined as in Section 3 and has grading $\deg(a[n]) = \deg(a)$. In other words, if $Y = \mathrm{Spec}(A)$ is the cone over X, then $\Lambda^0 X$ is the categorical quotient of $L^0(Y) - \{0\}$ by $\mathbb{C}^*$. The multiplication by t defines a self-embedding $\epsilon : L^0 Y \to L^0 Y$ which descends to an embedding $E : \Lambda^0 X \to \Lambda^0 X$. The ind-scheme

$$\Lambda X = \text{``}\varinjlim\text{``}\left\{\Lambda^0 X \xrightarrow{E} \Lambda^0 X \xrightarrow{E} ...\right\}$$

is an algebro-geometric analog of the $\mathbb{Z}$-cover of $\mathcal{L}X$ corresponding to $c_1(\mathcal{O}_X(1)) \in H^2(X, \mathbb{Z})$. The case of the universal cover can be treated in a similar way.

While $\Lambda^0 X$ is no longer smooth , it is nevertheless possible to define an analog of the Gysin maps and perform the limit construction. This is based on the Cartesian square

$$\begin{array}{ccc} L^0 Y & \xrightarrow{\epsilon} & L^0 Y \\ \downarrow & & \downarrow \\ \{0\} & \longrightarrow & Y \end{array}$$

If $d = \dim(X)$, we have a class $\xi \in H^{2d+2}_{\{0\}}(Y, \mathbb{C})$ coming from the fundamental class in $H^{2d}(X, \mathbb{C})$. From the above square we get a class $\eta \in H^{2d+2}_{\epsilon(L^0 Y)}(L^0 Y, \mathbb{C})$. The embedding E has codimension $2d + 2$ and η, being $\mathbb{C}^*$-equivariant, descends to

$$\zeta \in H^{2d+2}_{E(\Lambda^0 X)}(\Lambda^0 X, \mathbb{C}) = \mathrm{Hom}(E^! \underline{\mathbb{C}}, \underline{\mathbb{C}}[2d + 2]).$$

The map ζ provides a Gysin map in our non-smooth case. We define the partial Floer cohomology (corresponding to the given projective embedding) as

$$HF(\Lambda X) = \varinjlim\left\{H^\bullet(\Lambda^0 X, \mathbb{C}) \xrightarrow{E_!} H^{\bullet+2d+2}(\Lambda^0 X, \mathbb{C}) \xrightarrow{E_!} H^{\bullet+4d+4}(\Lambda^0 X, \mathbb{C}) \xrightarrow{E_!} ...\right\}.$$

In general, we do not expect a direct relation between $HQ(X)$ and the ring structure on $H^\bullet(\Lambda^0 X)$.

Note that the homotopy limit of the diagram of complexes of sheaves on $\Lambda^0 X(\mathbb{C})$

$$\underline{\mathbb{C}} \to E^! \underline{\mathbb{C}}[2d + 2] \to (E^2)^! \underline{\mathbb{C}}[4d + 4] \to ...$$

can be seen as a topological dualizing sheaf, cf. the discussion in [8] in a more restricted situation.

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Department of Mathematics, Yale University, New Haven CT 06520
E-mail address: `serguei.arkhipov@yale.edu, mikhail.kapranov@yale.edu`